Séminaire Lotharingien de Combinatoire **78B** (2017)
Article #9, 12 pp.

Proceedings of the 29th Conference on Formal Power Series and Algebraic Combinatorics (London)

Supercharacter theories of type A unipotent radicals and unipotent polytopes

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Abstract. Even with the introduction of supercharacter theories, the representation theory of many unipotent groups remains mysterious. This paper constructs a family of supercharacter theories for normal pattern groups in a way that exhibit many of the combinatorial properties of the set partition combinatorics of the full uni-triangular groups, including combinatorial indexing sets, dimensions, and computable character formulas. Associated with these supercharacter theories is also a family of polytopes whose integer lattice points give the theories geometric underpinnings.

Keywords: supercharacters, lattice polytopes, unipotent groups

1 Introduction

Supercharacter theory has infused the representation theory of unipotent groups with the combinatorics of set partitions. Specifically, set partitions index the supercharacters of the maximal unipotent upper-triangular subgroup UT of the finite general linear group GL [2, 13], and similar theories exist for the maximal unipotent subgroups of other finite reductive groups [3, 4]. However, while there are supercharacter theories for other unipotent groups, they do not generally exhibit this computable and combinatorial nature. This paper seeks to define a natural family of supercharacter theories for the normal pattern subgroups of UT. As an added bonus, we not only obtain a combinatorial description for these theories, but also gain geometric underpinnings coming from a family of lattice polytopes.

Diaconis–Isaacs defined a supercharacter theory of a finite group G as a direct analogue of its character theory, where they replacing conjugacy classes with superclasses and irreducible characters with supercharacters [9]. Their approach is based on André's adaption of the Kirillov orbit method to study UT, and the underlying axioms are calibrated to preserve as many properties of irreducible characters and conjugacy classes as possible. For example, the supercharacters are an orthogonal (but not generally orthonormal) basis for the space of functions that are constant on superclasses. This definition has given us new approaches to groups whose representation theories are known to be difficult (eg. unipotent groups). Not only can these new theories be combinatorially striking [1], but they can also be used in place of the usual character theory [5] in applications, they give a starting point in studying difficult theories [10], or give character theoretic foundations for number theoretic identities (eg. [6, 11]).

The supercharacter theories of this paper are fundamentally based on André's original construction for UT [2] and Diaconis–Isaacs' later generalization to algebra groups [9]. These constructions use two-sided orbits in the dual space $\mathfrak{ut}^*$ of the corresponding Lie algebra $\mathfrak{ut}$ of UT to construct the supercharacters. In the algebra group case the group UT acts on $\mathfrak{ut}^*$ by left and right multiplication (technically pre-composition by left and right multiplication on $\mathfrak{ut}$). In this paper we modify this construction by instead acting by parabolic subgroups of GL. The resulting theory is coarser but far more combinatorial in nature. In particular, we obtain statistics such as dimension, nestings and crossings that generalize the corresponding set partition statistics [8], and in Theorem 4.4 we give a character formulas with a "factorization" analogous to the well-known UT-cases.

For each supercharacter theory there is an associated polytope whose integer lattice points index the supercharacters of the theory. Thus, the supercharacter theories could in principle give a categorified version of the Ehrhart polynomials of these polytopes. These polytopes include all transportation polytopes [12], and may be viewed as a family of subfaces of transportation polytopes. This point of view not only gives a geometric approach to these supercharacter theories, but it also re-interprets set partitions as vertices of a polytope. Since I am unaware of other contexts where these polytopes may have been studied, I will refer to them as *unipotent polytopes*. At present we do not understand the significance of this geometry in the representation theory of unipotent groups, and this seems to be a promising direction for future work.

2 Preliminaries

This section reviews the relevant unipotent groups, a combinatorial interpretation of normality, and some of the standard supercharacter theories for these groups.

2.1 Normal pattern groups

Let $\mathcal{N}$ be a fixed total order of a finite set with N elements and fix a finite field $\mathbb{F}_q$ with q elements (eg. the total order $1 < 2 < \cdots < N$). Let $\mathrm{GL}_{\mathcal{N}}$ denote the finite general linear group on matrices with rows and columns indexed by our finite set in the order dictated by $\mathcal{N}$. If $\mathrm{char}(\mathbb{F}_q) = p$, then a Sylow p-subgroup of $\mathrm{GL}_{\mathcal{N}}$ is the subgroup of unipotent upper-triangular matrices

$$\mathrm{UT}_{\mathcal{N}} = \{g \in \mathrm{GL}_{\mathcal{N}} \mid (g - \mathrm{Id}_N)_{ij} \neq 0 \text{ implies } i <_{\mathcal{N}} j\}, \quad \text{and} \quad \mathfrak{ut}_{\mathcal{N}} = \mathrm{UT}_{\mathcal{N}} - \mathrm{Id}_N$$

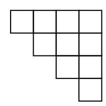
is a corresponding nilpotent $\mathbb{F}_q$-algebra. If $\mathfrak{n} \subseteq \mathfrak{ut}_{\mathcal{N}}$ is any subalgebra, then we obtain a subgroup $\mathrm{Id}_N + \mathfrak{n} \subseteq \mathrm{UT}_{\mathcal{N}}$ called and *algebra subgroup*. If $\mathcal{P}$ is a subposet of $\mathcal{N}$ on the same underlying set, then we call the algebra subgroup

$$\mathrm{UT}_{\mathcal{P}} = \mathrm{Id}_N + \mathfrak{ut}_{\mathcal{P}} \subseteq \mathrm{UT}_{\mathcal{N}}, \qquad \text{where} \qquad \mathfrak{ut}_{\mathcal{P}} = \{x \in \mathfrak{ut}_{\mathcal{N}} \mid x_{ij} \neq 0 \text{ implies } i \prec_{\mathcal{P}} j\},$$

a *pattern subgroup* of $\mathrm{UT}_{\mathcal{N}}$. Note that transitivity in the poset $\mathcal{P}$ exactly implies that $\mathrm{UT}_{\mathcal{P}}$ is closed under multiplication.

In general, a subposet $\mathcal{P}$ of a poset $\mathcal{Q}$ does not give a normal subgroup $\mathrm{UT}_{\mathcal{P}}$ of $\mathrm{UT}_{\mathcal{Q}}$. However, there is a straight-forward condition on the poset that characterizes this group theoretic condition: a subposet $\mathcal{P} \subseteq \mathcal{Q}$ is *normal* if $j \prec_{\mathcal{P}} k$ implies $i \prec_{\mathcal{P}} k$ and $j \prec_{\mathcal{P}} l$ for all $i \prec_{\mathcal{Q}} j$ and $k \prec_{\mathcal{Q}} l$. In this case, we write $\mathcal{P} \lhd \mathcal{Q}$.

There are a number of combinatorial interpretations of normal posets of the total order $\mathcal{N}$. For $N \in \mathbb{Z}_{\geqslant 0}$, let D_N denote the Young diagram $(N-1, N-2, \ldots, 1)$, where we right justify the rows. For example,

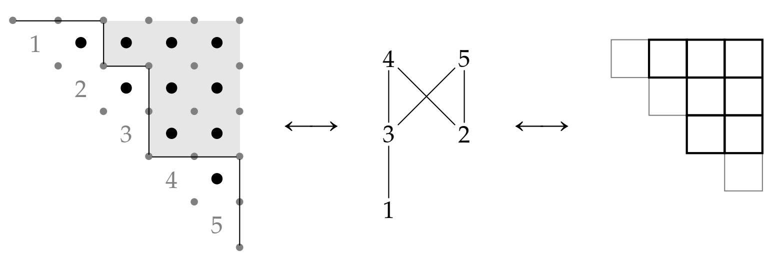
$D_5 =$  .

Proposition 2.1. *There are bijections*

$$\begin{array}{ccccc} \left\{\begin{matrix}\text{Dyck paths from}\\ (0,0) \text{ to } (2N,-2N)\end{matrix}\right\} & \longleftrightarrow & \left\{\begin{matrix}\text{normal sub-}\\ \text{posets of } \mathcal{N}\end{matrix}\right\} & \longleftrightarrow & \left\{\begin{matrix}\text{sub-Young}\\ \text{diagrams of } D_N\end{matrix}\right\} \\ d_{\mathcal{P}} & \longleftarrow\!\shortmid & \mathcal{P} & \longmapsto & F_{\mathcal{P}}. \end{array}$$

where $(2i-1, 2j-1)$ *is NorthEast of* $d_{\mathcal{P}}$ *if and only if* $i \prec_{\mathcal{P}} j$ *if and only if* $(i,j) \in F_{\mathcal{P}}$.

Example. For example, if $2N = 10$, then



where the shaded region accentuates the relevant points NorthEast of the Dyck path.

2.2 Supercharacter theories of unipotent groups

Supercharacter theories for finite groups were first defined in [9], generalizing work by André studying representations of $\mathrm{UT}_{\mathcal{N}}$ (a series of papers starting with [2]). There are numerous equivalent formulations of a supercharacter theory, but the following seems most suitable for the purposes of this paper.

A *supercharacter theory* $(\mathcal{K}, \mathcal{X})$ of a finite group G is a pair, where $\mathcal{K}$ is a partition of G and $\mathcal{X}$ is a set of characters, such that

(SC0) The number of blocks of $\mathcal{K}$ is the same as the number of elements in $\mathcal{X}$.

(SC1) Each block $K \in \mathcal{K}$ is a union of conjugacy classes.

(SC2) The set
$$\mathcal{X} \subseteq \{\theta : G \to \mathbb{C} \mid \theta(g) = \theta(h), g, h \in K, K \in \mathcal{K}\}.$$

(SC3) Each irreducible character of G is the constituent of exactly one element in $\mathcal{X}$.

We refer to the blocks of $\mathcal{K}$ as *superclasses* and the elements of $\mathcal{X}$ as *supercharacters*.

While we have many ways of constructing supercharacter theories, general constructions are not well-understood. That is, given a finite groups, it is a hard problem to determine its supercharacter theories. Some groups have remarkably few supercharacter theories, such as the symplectic group $\mathrm{Sp}_6(\mathbb{F}_2)$ with exactly 2 [7], and some groups have surprisingly many, such as $C_3 \times C_6$ with 297 distinct supercharacter theories. However, for this paper we follow the basic strategy laid out by [9] for algebra groups.

Let $\mathrm{Id}_N + \mathfrak{n} \subseteq \mathrm{UT}_{\mathcal{N}}$ be an algebra subgroup. Then $\mathrm{Id}_N + \mathfrak{n}$ acts on both $\mathfrak{n}$ and its vector space dual $\mathfrak{n}^*$ by left and right multiplication, where

$$(a \cdot y \cdot b)(x) = y(a^{-1}xb^{-1}), \qquad \text{for } a, b \in \mathrm{Id}_N + \mathfrak{n},\ x \in \mathfrak{n},\ y \in \mathfrak{n}^*.$$

Fix a nontrivial homomorphism $\vartheta : \mathbb{F}_q^+ \to \mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^\times$. In this situation [9] define a supercharacter theory given by

AG**-superclasses.** The set partition $\{\mathrm{Id}_N + (\mathrm{Id}_N + \mathfrak{n})x(\mathrm{Id}_N + \mathfrak{n}) \mid x \in \mathfrak{n}\}$ of $\mathrm{Id}_N + \mathfrak{n}$.

AG**-supercharacters.** The set of characters

$$\left\{\chi_{AG}^{y} = \sum_{z \in (\mathrm{Id}_N + \mathfrak{n})y(\mathrm{Id}_N + \mathfrak{n})} \vartheta \circ z \mid y \in \mathfrak{n}^*\right\}.$$

Remark. In the case where $\mathfrak{n} = \mathfrak{ut}_{\mathcal{N}}$, this supercharacter theory gives a nice combinatorial theory developed algebraically by André [2] and more combinatorially by Yan [13]. However, in general even this supercharacter theory may be wild for algebra subgroups. In fact, we do not even understand it for pattern subgroups.

3 Block normal pattern subgroups

In this section, we build a family of pattern subgroups of $\mathrm{UT}_{\mathcal{N}}$; they will all be normal, and each will have a family of supercharacter theories, defined in Section 4. The choice of a subgroup with an associated supercharacter theory will determine a polytope, giving the theory a geometric foundation.

3.1 Parabolic posets and UT_β

We begin by defining a family of unipotent groups that appear naturally in the theory of reductive groups, the unipotent radicals of parabolic subgroups. It turns out that for $\mathrm{GL}_{\mathcal{N}}$, these unipotent groups are pattern groups and their associated posets are easy to characterize. In Section 4, each unipotent radical UT_β will determine a family of supercharacter theories.

A subposet $\mathcal{Q}$ is *parabolic* in $\mathcal{N}$ if there exists a set composition $(\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_\ell)$ of the underlying set such that $a \prec_{\mathcal{Q}} b$ if and only if $a \in \mathcal{Q}_i$ and $b \in \mathcal{Q}_j$ with $i < j$. These subposets are necessarily normal, where the corresponding Dyck path always returns down to the diagonal before moving right again. We will write $\mathcal{Q} \lhd_{\mathrm{pb}} \mathcal{N}$.

Since given a total order $\mathcal{N}$ the sizes of the blocks of the set composition completely determines $\mathcal{Q}$, we deduce the following proposition.

Proposition 3.1. *There is a bijection*

$$\begin{array}{rccc} \mathrm{bdry}: & \left\{\begin{matrix}\textit{Parabolic sub-}\\ \textit{posets of } \mathcal{N}\end{matrix}\right\} & \longrightarrow & \left\{\begin{matrix}\textit{integer com-}\\ \textit{positions of } N\end{matrix}\right\} \\ & \mathcal{Q} & \mapsto & (|\mathcal{Q}_1|, \dots, |\mathcal{Q}_\ell|). \end{array}$$

For an integer composition $\beta \vDash N$ and an underlying total order $\mathcal{N}$, define

$$\mathrm{UT}_\beta = \mathrm{UT}_{\mathrm{bdry}^{-1}(\beta)}$$

(note that $\mathrm{bdry}^{-1}(\beta)$ makes no sense without $\mathcal{N}$).

Every parabolic subposet $\mathcal{Q}$ in $\mathcal{N}$ with $\beta = \mathrm{bdry}(\mathcal{Q})$ has a corresponding Levi subgroup

$$L_\beta = \left[\begin{array}{c|c|c|c} \mathrm{GL}_{\beta_1} & 0 & \cdots & 0 \\ \hline 0 & \mathrm{GL}_{\beta_2} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & \mathrm{GL}_{\beta_\ell} \end{array}\right],$$

such that UT_β is the unipotent radical of the parabolic subgroup

$$P_\beta = L_\beta \ltimes \mathrm{UT}_\beta = N_{\mathrm{GL}_{\mathcal{N}}(q)}(\mathrm{UT}_\beta).$$

Remark. The Lie theoretic language of parabolics, Levis and unipotent radicals is merely given for context. The reader is welcome to ignore the terminology and focus on the definitions.

3.2 Unipotent polytopes and $\mathrm{UT}_{(\beta,\mathcal{P})}$

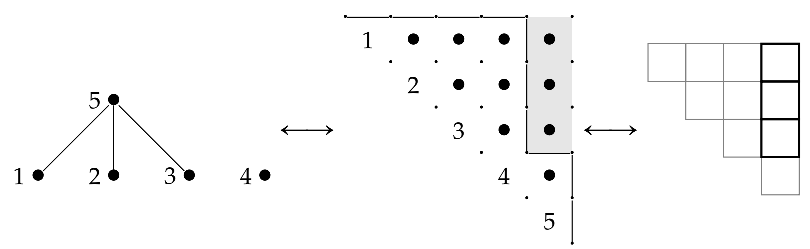
Fix an integer composition $\beta = (\beta_1, \ldots, \beta_\ell) \models N$ and let $\mathcal{P}$ be a normal subposet of $1 < 2 < \cdots < \ell$ with corresponding Ferrers shape F. The *unipotent polytope* $(\beta, \mathcal{P})$ is the convex polytope in the positive quadrant $\mathbb{R}^{|F|}_{\geqslant 0}$ determined by the inequalities

$$\Big\{ \sum_{i \prec_{\mathcal{P}} j} x_{ij} \leqslant \beta_j, \sum_{j \prec_{\mathcal{P}} k} x_{jk} \leqslant \beta_j \;\Big|\; 1 \leqslant j \leqslant \ell \Big\}.$$

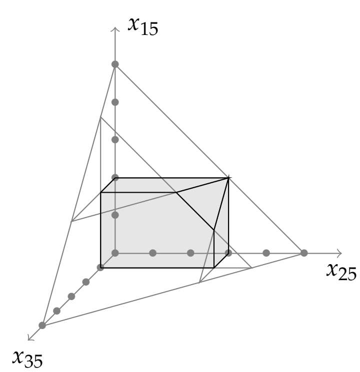
Remark. If F is the Ferrers shape corresponding to $\mathcal{P}$, then one may view the unipotent polytope as possible fillings of the boxes of F by non-negative real numbers such that the row and column sums are bounded by β.

Examples.

(E1) If $\beta = (2, 3, 1, 1, 5)$, and

$\mathcal{P} =$ 

then the equations $x_{15} \leqslant 2, x_{25} \leqslant 3, x_{35} \leqslant 1, x_{15} + x_{25} + x_{35} \leqslant 5$ give the polytope



(E2) In general, the unipotent polytopes $(\beta, \mathcal{P})$ where the corresponding Ferrers shape $F_{\mathcal{P}}$ is a square are transportation polytopes (with a few additional faces). They are a subfamily corresponding to abelian unipotent groups. In this case, the bounds on the row sums and and the bounds on the column sums are independent.

If $\mathrm{bdry}^{-1}(\beta)$ has corresponding set composition $(\mathcal{Q}_1, \ldots, \mathcal{Q}_\ell)$, then a unipotent polytopes $(\beta, \mathcal{P})$ determines a subgroup

$$\mathrm{UT}_{(\beta,\mathcal{P})} = \mathrm{UT}_{\mathrm{fat}_\beta(\mathcal{P})} \lhd \mathrm{UT}_\beta, \tag{3.1}$$

where $\mathrm{fat}_\beta(\mathcal{P})$ is the subposet of $\mathrm{bdry}^{-1}(\beta)$ given by

$$a \prec_{\mathrm{fat}_\beta(\mathcal{P})} b \qquad \text{if and only if} \qquad a \in \mathcal{Q}_i,\, b \in \mathcal{Q}_j \text{ with } i \prec_{\mathcal{P}} j.$$

Remark. For a fixed total order $\mathcal{N}$ with N elements, the function

$$\begin{array}{ccc} \left\{\begin{matrix}\text{unipotent polytopes}\\ (\beta,\mathcal{P}) \text{ with } |\beta| = N\end{matrix}\right\} & \longrightarrow & \left\{\begin{matrix}\text{normal sub-}\\ \text{groups of } \mathrm{UT}_{\mathcal{N}}\end{matrix}\right\} \\ (\beta,\mathcal{P}) & \mapsto & \mathrm{UT}_{(\beta,\mathcal{P})} \end{array}$$

is not injective, since, for example,

$$\mathrm{UT}_{\left((1^4),\, \begin{smallmatrix}3 & & 4\\ | & \times & |\\ 1 & & 2\end{smallmatrix}\right)} = \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathrm{UT}_{\left((2,2),\, \begin{smallmatrix}2\\ |\\ 1\end{smallmatrix}\right)}.$$

On the other hand, for a fixed $\beta \models N$, the function is injective (as $\mathcal{P}$ varies).

4 Parabolic supercharacter theories

The data in a unipotent polytope $(\beta, \mathcal{P})$ also gives a natural supercharacter theory to a corresponding unipotent group $\mathrm{UT}_{(\beta,\mathcal{P})}$. This section defines this theory, shows that the supercharacters/superclasses are indexed by the integer lattice points contained in the polytope $(\beta, \mathcal{P})$, and then gives character formulas.

4.1 Supercharacter theories and their index sets

Let $(\beta, \mathcal{P})$ be a unipotent polytope with $\beta \models N$. Then, as before, we have

$$\mathfrak{ut}_{(\beta,\mathcal{P})} = \mathrm{UT}_{(\beta,\mathcal{P})} - \mathrm{Id}_N \quad \text{and} \quad \mathfrak{ut}^*_{(\beta,\mathcal{P})} = \mathrm{Hom}_{\mathbb{F}_q}(\mathfrak{ut}_{(\beta,\mathcal{P})}, \mathbb{F}_q).$$

The group P_β acts on both $\mathfrak{ut}_{(\beta,\mathcal{P})}$ and $\mathfrak{ut}^*_{(\beta,\mathcal{P})}$ by left and right multiplication, where

$$(a \cdot y \cdot b)(x) = y(a^{-1}xb^{-1}), \quad \text{for } x \in \mathfrak{ut}_{(\beta,\mathcal{P})}, y \in \mathfrak{ut}^*_{(\beta,\mathcal{P})}, a, b \in P_\beta.$$

These actions give a natural supercharacter theory for $\mathrm{UT}_{(\beta,\mathcal{P})}$.

P_β-superclasses of $\mathrm{UT}_{(\beta,\mathcal{P})}$. The set partition $\{P_\beta x P_\beta + \mathrm{Id}_N \mid x \in \mathfrak{ut}_{(\beta,\mathcal{P})}\}$ of $\mathrm{UT}_{(\beta,\mathcal{P})}$.

P_β-supercharacters of $\mathrm{UT}_{(\beta,\mathcal{P})}$. The characters

$$\{\chi^y_\beta \mid P_\beta y P_\beta \in P_\beta \backslash \mathfrak{ut}^*_{(\beta,\mathcal{P})} / P_\beta\}, \quad \text{where} \quad \chi^y_\beta = \sum_{z \in P_\beta y P_\beta} \vartheta \circ z. \tag{4.1}$$

Proposition 4.1. *If $(\beta, \mathcal{P})$ is a unipotent polytope, then the P_β-superclasses and the P_β-supercharacters form a supercharacter theory of* $\mathrm{UT}_{(\beta,\mathcal{P})}$.

Remark. For $y \in \mathrm{ut}^*_{(\beta,\mathcal{P})}$, define the $\mathrm{UT}_{(\beta,\mathcal{P})}$-module M^y by a $\mathbb{C}$-basis

$$\{\boxed{z} \mid z \in P_\beta y P_\beta\}$$

with an action

$$u \cdot \boxed{z} = \vartheta \circ z(u^{-1} - \mathrm{Id}_{|\beta|})\boxed{uz} \qquad \text{for } u \in \mathrm{UT}_{(\beta,\mathcal{P})},\, z \in P_\beta y P_\beta.$$

The trace of M^y is the supercharacter χ^y_β.

For a unipotent polytope $(\beta, \mathcal{P})$ with $F_\mathcal{P}$ as in Proposition 2.1, let

$$\mathcal{T}^\beta_\mathcal{P} = \left\{ \begin{array}{rcl} \lambda : F_\mathcal{P} & \to & \mathbb{Z}_{\geqslant 0} \\ (i,j) & \mapsto & \lambda_{ij} \end{array} \;\middle|\; \sum_{\substack{k \\ (j,k)\in F_\mathcal{P}}} \lambda_{jk}, \sum_{\substack{i \\ (i,j)\in F_\mathcal{P}}} \lambda_{ij} \leqslant \beta_j, 1 \leqslant j \leqslant \ell \right\}$$

be the set of $\mathbb{Z}_{\geqslant 0}$-lattice points contained in or on $(\beta, \mathcal{P})$. The following theorem establishes the connection between P_β-supercharacter theories and $\mathbb{Z}_{\geqslant 0}$-lattice points in unipotent polytopes.

Theorem 4.2. *For $(\beta, \mathcal{P})$ a unipotent polytope,*

$$\left\{ \begin{array}{c} P_\beta\text{-superclasses} \\ \text{of } \mathrm{UT}_{(\beta,\mathcal{P})} \end{array} \right\} \longleftrightarrow \mathcal{T}^\beta_\mathcal{P} \longleftrightarrow \left\{ \begin{array}{c} P_\beta\text{-supercharacters} \\ \text{of } \mathrm{UT}_{(\beta,\mathcal{P})} \end{array} \right\}.$$

Examples.

(E1) The set

$$\mathcal{T}^{(4,1,2)}_{\substack{3 \\ 1\ \ 2}} = \left\{ \begin{array}{ccc} 4 & & \boxed{0} \\ & 1 & \boxed{0} \\ & & 2 \end{array}, \begin{array}{ccc} 4 & & \boxed{1} \\ & 1 & \boxed{0} \\ & & 2 \end{array}, \begin{array}{ccc} 4 & & \boxed{0} \\ & 1 & \boxed{1} \\ & & 2 \end{array}, \begin{array}{ccc} 4 & & \boxed{1} \\ & 1 & \boxed{1} \\ & & 2 \end{array}, \begin{array}{ccc} 4 & & \boxed{2} \\ & 1 & \boxed{0} \\ & & 2 \end{array}, \right\}$$

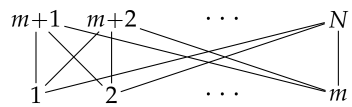
where the entries shaded in gray give the bounds for each row and column.

(E2) If $\beta = (1^m, n)$ with $m \leqslant n$ and

$$\mathcal{P} = \begin{array}{cccc} & m+1 & & \\ 1 & 2 & \cdots & m \end{array}$$

then $\mathcal{T}^\beta_\mathcal{P}$ is the set of vertices of the m-dimensional hypercube.

(E3) If $N = 2m$, and

$\mathcal{P} =$ 

then $\mathcal{T}_{\mathcal{P}}^{(1^N)}$ is the usual basis for the rook monoid.

(E4) If $\mathcal{N}$ is a total order of a set A, then the set $\mathcal{T}_{\mathcal{N}}^{(1^N)}$ is in bijection with the set of set partitions of A.

4.2 Supercharacter formulas

There are a number of statistics that arise naturally in the P_β-supercharacter theories. They naturally generalize their set partition analogues in the $P_{(1^N)}$-supercharacter theory of $\mathrm{UT}_{\mathcal{N}}$ (see [8] for a more general algebraic framework for these statistics).

For $\lambda \in \mathcal{T}_{\mathcal{P}}^{\beta}$ with $\mathcal{Q}$ the usual linear order on $\{1, 2, \ldots, \ell(\beta)\}$, there are a number of ways to measure the "size" of a λ. For example,

$$|\lambda| = \sum_{i \prec_{\mathcal{P}} j} \lambda_{ij}$$

measures the lattice distance to the origin of the lattice point in the unipotent polytope. However, geometric interpretations of the other statistics are unknown (at least to me). Having more to do with the dimension of the corresponding modules,

$$\dim_L(\lambda) = \sum_{i \prec_{\mathcal{P}} j \prec_{\mathcal{Q}} k} \lambda_{ik}\beta_j \qquad \text{and} \qquad \dim_R(\lambda) = \sum_{i \prec_{\mathcal{Q}} j \prec_{\mathcal{P}} k} \lambda_{ik}\beta_j$$

give the *left* and *right dimensions* of λ (respectively). Note that if $\mathcal{P} = \mathcal{Q}$, then $\dim_R(\lambda) = \dim_L(\lambda)$. To account for over-counting, we also require the *crossing number*

$$\mathrm{crs}(\lambda) = \sum_{i \prec_{\mathcal{Q}} j \prec_{\mathcal{P}} k \prec_{\mathcal{Q}} l} \lambda_{ik}\lambda_{jl}$$

of λ. Lastly, if $\mu \in \mathcal{T}_{\mathcal{P}}^{\beta}$, the *nestings* of μ in λ are

$$\mathrm{nst}_{\mu}^{\lambda} = \sum_{i \prec_{\mathcal{P}} j \prec_{\mathcal{P}} k \prec_{\mathcal{P}} l} \lambda_{il}\mu_{jk}.$$

Example. if $\beta = (3, 6, 3, 4, 5, 1)$,

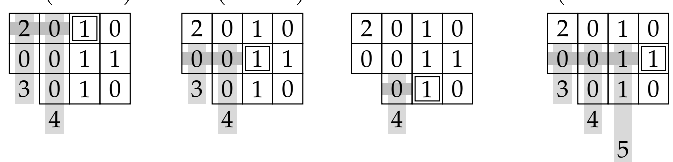
$$\lambda = \begin{array}{cccccc} 3 & & 2 & 0 & 1 & 0 \\ & 6 & 0 & 0 & 1 & 1 \\ & & 3 & 0 & 1 & 0 \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 1 \end{array} \qquad \text{and} \qquad \mu = \begin{array}{cccccc} 3 & & 0 & 1 & 0 & 1 \\ & 6 & 1 & 0 & 2 & 0 \\ & & 3 & 1 & 2 & 0 \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & 1 \end{array}$$

then

$$|\lambda| = 6\cdot 0 + 4\cdot 1 + 1\cdot 2$$

$$\qquad 1\cdot(3+4) \;+\; 1\cdot(3+4) \;+\; \quad 1\cdot 4 \quad +\; 1\cdot(3+4+5)$$

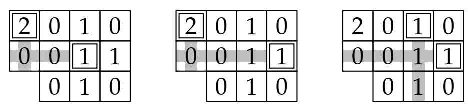
$\dim_L(\lambda) =$



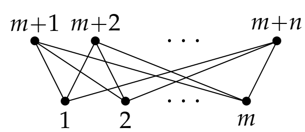
$$\dim_R(\lambda) = 2\cdot 6 + 1\cdot(6+3) + 1\cdot 3 + 1\cdot 3$$
$$\mathrm{nst}^{\lambda}_{\mu} = 1\cdot 1 + 3\cdot 1\cdot 1 + 2\cdot 1$$

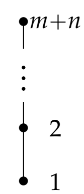
$$2\cdot 1 \quad + \quad 2\cdot 1 \quad + \quad 1\cdot 1$$

$\mathrm{crs}(\lambda) =$



The most basic case for us is when the polytope is a line segment, so consider the case where

$\beta = (m,n), \quad \mathcal{Q} = \mathrm{bdry}^{-1}(\beta) =$ 

for $\mathcal{N} =$ 

.

Then

$$\mathrm{UT}_{(\beta,\,\bullet\!-\!\bullet)} = \left\{ \left[\begin{array}{c|c} \mathrm{Id}_m & A \\ \hline 0 & \mathrm{Id}_n \end{array}\right] \;\middle|\; A \in M_{m\times n}(\mathbb{F}_q) \right\} \cong (\mathbb{F}_q^+)^{mn}.$$

In this case, the superclass of

$$\left[\begin{array}{c|c} \mathrm{Id}_m & A \\ \hline 0 & \mathrm{Id}_n \end{array}\right] \quad \text{is determined by} \quad \mathrm{rank}(A) \in \mathcal{T}^{(m,n)}_{\bullet\!-\!\bullet} = \{0,1,\ldots,\min\{m,n\}\}.$$

For $n,k \in \mathbb{Z}_{\geqslant 0}$, let

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}, \qquad \text{where} \qquad [n]! = [n][n-1]\cdots[2][1] \quad \text{and} \quad [n] = \frac{q^n-1}{q-1}.$$

Theorem 4.3. *If* $0 \leqslant j,l \leqslant \min\{m,n\}$ *and* $u_{(j)} = \mathrm{Id}_{m+n} + e_{(j)} \in \mathrm{UT}_{(\beta,\mathcal{Q})}$, *then*

$$\chi^{(l)}_{(m,n)}(u_{(j)}) = \sum_{\substack{a,b\in\mathbb{Z}_{\geqslant 0}\\ a+b=l}} (-1)^a q^{bj+\binom{a}{2}} \begin{bmatrix} j \\ a \end{bmatrix}_q \chi^{(b)}_{(m-j,n-j)}(u_{(0)}),$$

where

$$\chi^{(l)}_{(m,n)}(u_{(0)}) = |\mathrm{GL}_l(\mathbb{F}_q)| \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q = \#\left\{ \begin{matrix} m\times n \text{ matrices} \\ \text{of rank } l \end{matrix} \right\}.$$

Remark. We use the convention that $|\mathrm{GL}_0(\mathbb{F}_q)| = 1$.

Let $(\beta, \mathcal{P})$ be an arbitrary unipotent polytope. For $\lambda, \mu \in \mathcal{T}_{\mathcal{P}}^{\beta}$, let

$$\begin{aligned} \mathrm{loc}_{\mu}^{\lambda} : \{(j,l) \mid j \prec_{\mathcal{P}} l\} &\longrightarrow \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \\ (j,l) &\mapsto \Big(\beta_j - \sum_{j \prec_{\mathcal{P}} k \prec_{\mathcal{P}} l} \mu_{jk} - \sum_{l \prec_{\mathcal{P}} m} \lambda_{jm}, \beta_l - \sum_{j \prec_{\mathcal{P}} k \prec_{\mathcal{P}} l} \mu_{kl} - \sum_{i \prec_{\mathcal{P}} j} \lambda_{il}\Big). \end{aligned}$$

For example, if $\beta = (3,6,3,4,5,1)$,

$$\lambda = \begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & 1 & 0 \\ \cline{2-4} \end{array} \qquad \text{and} \qquad \mu = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 2 & 0 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 0 \\ \cline{2-4} \end{array}$$

then

$$\begin{aligned} \mathrm{loc}_{\mu}^{\lambda}(2,5) &= \left(\beta_2 - \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & \boxed{2} & 0 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 0 \\ \cline{2-4} \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 0 \\ \hline 0 & 0 & \boxed{1} & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & 1 & 0 \\ \cline{2-4} \end{array}, \beta_5 - \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline 1 & 0 & \boxed{2} & 0 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 & 0 \\ \cline{2-4} \end{array} - \begin{array}{|c|c|c|c|} \hline 2 & 0 & 1 & 0 \\ \hline 0 & 0 & \boxed{1} & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & 1 & 0 \\ \cline{2-4} \end{array}\right) \\ &= (6 - (1+0) - 1, 5 - 2 - 1) \\ &= (4,2). \end{aligned}$$

Theorem 4.4. *For $\lambda, \mu \in \mathcal{T}_{\mathcal{P}}^{\beta}$ and $u_\mu = 1 + e_\mu$,*

$$\chi_{\beta}^{\lambda}(u_\mu) = \frac{q^{\dim_L(\lambda) + \dim_R(\lambda)}}{q^{\mathrm{nst}_{\mu}^{\lambda} + \mathrm{crs}(\lambda)}} \prod_{j \prec_{\mathcal{P}} l} \chi_{\mathrm{loc}_{\mu}^{\lambda}(j,l)}^{(\lambda_{jl})}(u_{(\mu_{jl})}).$$

Acknowledgements

The author would like to thank H. Hausmann for pointing out the connection between the combinatorics of bounded row and column sums and integer polytopes, and to K. Meszaros for directing me to transportation polytopes. Also, thanks to M. Aguiar for (a) helping work out the corresponding monoid structure and (b) convincing me it is too involved for this paper. The work was also partially supported by NSA H98230-13-0203.

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